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Betchov, Robert; Larsen, Poul Scheel

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A non-Gaussian model of turbulence (soccer-ball integrals)

Robert Betchov

Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, Indiana 46556

P. S. Larsen

Department of Fluid Mechanics, Technical University of Denmark, DK-2800 Lyngby, Denmark
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The statistics of the time evolution of a nonlinearly coupled system of first-order equations representing the Euler equations is studied. The probability distribution of functions is nearly Gaussian, while that of their time derivatives has exponential tails and moments of order 4, 6, and 8 that approach those of the exponential distributions.

Since the early work of Townsend[1] we know that the velocity fluctuations of a turbulent flow have Gaussian probability distributions, but that the derivatives of velocity components, in space or in time, do not. Further work[2-4] showed that the probability distributions of derivatives have exponential tails. This is a distinct property, not directly related to the skewness factor, which reflects the asymmetry of probability distributions.

The Euler equations, which govern all but the smallest details of a turbulent flow, can, in many instances, be treated as a set of N first-order quadratic ordinary differential equations, where N is large. The general form is

$$\frac{dF_i(t)}{dt} = \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk} F_j(t) F_k(t) . \tag{1}$$

In wavenumber space, the Fourier transforms of the Euler equations takes this form.[5] It follows that the number of nonzero coefficients C_{ijk} in each equation is large and of the order of N. In physical space, a numerical computation involves the definition of a large number of grid points and spatial derivatives are expressed in terms of finite differences. The pressure is given by an integral over the entire volume. Thus, the Euler equations again lead to a first-order system such as Eq. (1) with N very large. Note that the coefficients always have the property that the energy $E = \sum_{i=1}^{N} F_i F_i$ is an invariant.

In incompressible flows the pressure depends formally upon the entire velocity field. However, the pressure gradients intervene mostly as local custodians of the conservation of mass, and pressure forces may be approximated by local velocity components.[6,7] This is equivalent to the use of a truncated Green's function in the solution of a Poisson equation.

With this approximation, the number of terms on the right-hand side of Eq. (1) is no longer proportional to N. In fact, only a handful of terms remain, corresponding to transport and local processes. This has profound statistical implications.

The present study examines the properties of systems for which the functions $F(t)$ have Gaussian probability distributions, but the derivatives do not. We noticed that the product of two statistically independent Gaussian functions has a probability distribution given by a Hankel function of order zero. This has two symmetric exponential tails and, at the origin, a mild logarithmic singularity. The sum of several such random Hankelians gently approaches a Gaussian around the origin. But, the exponential tails persist and only vanish very slowly under the effects of the central limit theorem.

We studied a numerical model of 60 functions such that each equation has only three bilinear terms. The model was later extended to $N = 120$, with six interacting terms. The geometric pattern of twelve pentagons and twenty hexagons, stitched together to form a soccer ball, suggests a simple, closed and highly symmetric set of coefficients C_{ijk}. Label the 60 corners in any sequence, and define each $F(t)$ as a quantity stored at one particular corner. Let the rate-of-change of any particular function be governed by the products of nearby functions, with the conservation of sum of squares.

The case of only three functions $X(t)$, $Y(t)$, and $Z(t)$, an elementary system corresponding to a classic elliptical integral, is well-known,

$$\frac{dX}{dt} = 2YZ, \quad \frac{dY}{dt} = -ZX, \quad \frac{dZ}{dt} = -XY . \tag{2}$$

Selecting this scheme to represent the interactions between points i, a, and b of Fig. 1, we define similar interactions between triplets of nearby corners on the soccer ball, yielding the following complete equation governing the function F_i

$$\frac{dF_i}{dt} = 2F_a F_b - F_p F_q - F_r F_s . \tag{3}$$

Note that we have only three terms on the right-hand side. Similar equations apply to all sixty functions. In particular, the equations for F_a and F_b also have

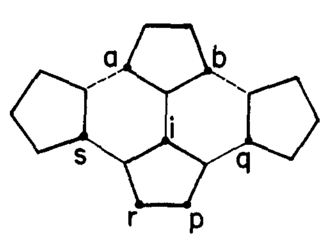

FIG. 1. Map of interactions contributing to the rate-of-change at point i, Eq. (3).

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three terms each, including a term $-F_b F_t$ and a term $-F_a F_t$, respectively. Thus, the interaction between the triplet F_t, F_a, and F_b will not alter the total energy of the system.

At time $t=0$, we choose random initial values for all the functions, normalized to a total energy of 60 units. Integration by the Runge–Kutta method, with $dt = 1/32$, in single precision, requires almost no renormalization to compensate for the slow accumulation of errors. Since the sixty functions have the same statistical properties, various correlations and probability distributions can easily be obtained.

The probability distribution of F is nearly Gaussian. The major deviation is related to the fact that, even when all functions are zero except one, the deviation is limited to $\sqrt{60}$ or about 7 standard deviations. The probability distribution of the derivative has exponential tails, and the moments of order 4, 6, and 8 approach those of the exponential distributions.

In order to improve on this result, we defined a system of 60 complex functions, equivalent to a set of $N = 120$ real functions. The only modification is that every function on the right-hand side of Eq. (3) must be replaced by its complex conjugate, in order to preserve the energy. This defines six real terms on the right-hand side of each of 120 real equations. The ceiling is now raised to about 11 standard deviations σ, and the agreement with the expected properties becomes excellent.

Some typical results are shown in Figs. 2 and 3, with scales suitable for stressing the aspects mentioned here. We integrated the system forward in time, up to $t = 6000$, in about 2×10^5 time steps. We determined the moments of order 1 through 8 for a sample of 5×10^5 data points, taken at intervals of 4 time steps. The results are shown in Table I, with column G for the theoretical Gaussian values, column F for the numerically obtained integrals, and column DF for their derivatives. The last line gives the nondimensional ratio involving only moments of order 8, 6, and 4 such that exponential distributions will have $R = 56/30 = 1.867$.

The results of Table I can be compared to experimental data obtained at high Reynolds numbers[3] when the instruments furnish a signal equal to the difference between successive values of a velocity component, at times t and $t+h$. When h is large, the signal is analogous to $F(t)$, and it shows nearly Gaussian properties with the ratio R near 1.38. When h is small, the signal is analogous to dF/dt and the even moments take values such as 1, 4.9, 39, 550. The ratio $R = 1.78$ confirms the presence of exponential tails. The comparison cannot apply to odd-moments since our model has no properties corresponding to the irreversibility of turbulent energy transfer or to spatial inhomogeneity.

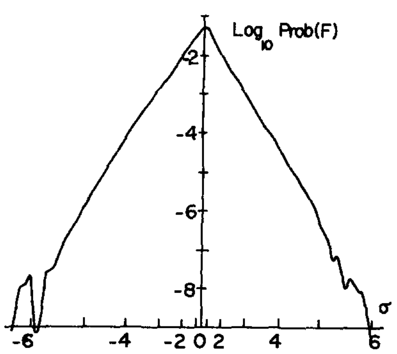

FIG. 2. Probability distribution of $F(t)$, $N = 120$. A symmetric triangle indicates a Gaussian.

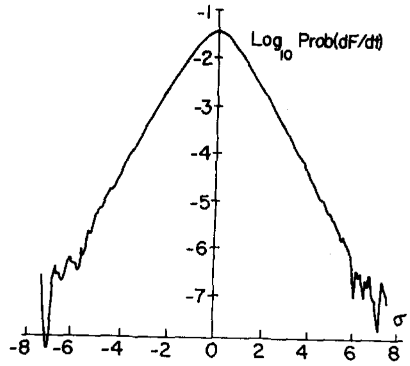

FIG. 3. Probability distribution of $dF(t)/dt$, $N = 120$. A symmetric triangle indicates an exponential.

Finally, the stability of the system, say the case of 60 real functions, deserves comment. If the energy is almost equally distributed between all regions of the soccer ball, an occasional perturbation is unimportant. If all functions are zero except one, say X, all derivatives will remain zero. One single perturbation in Y or Z can then excite an oscillation of the elliptic type, following Eq. (2). This oscillating triplet will not lose its energy, unless other perturbations disturb some nearby functions.

In general, the energy can be invested in several isolated regions, and if they are separated by bands of exact zeros, they will not interact. However, if the system is then exposed to any small level of random excitation, these islands will spread and merge into a statistically homogeneous solution. In a few preliminary numerical experiments we added some simple linear feedback terms, such that energy was slowly removed from one "polar" cap and randomly re-invested in the opposite cap. In the presence of a steady level of perturbations the linear process, feeding energy out of one part of the system and resupplying the opposite part, leads to a stormy type of solution. Indeed, small perturbations will trigger intermittent discharges of

TABLE I. Moments of distributions.

Order	G	F	DF
1	0	0.00	0.00
2	1	1.00	1.00
3	0	0.00	0.00
4	3	2.95	4.21
5	0	−0.01	0.10
6	15.	14.30	38.57
7	0.	−0.40	3.12
8	105	94.87	613.99
R	1.40	1.375	1.74

energy toward the damped regions.

Thus, it seems that the addition of linear terms, analogous to viscous damping forces, may render the system essentially sensitive to externally generated perturbations. Also, if the terms tend to drive the system to one of its singular solutions, the system will enhance its basic sensitivity to a constant background agitation. This suggests that linear terms and noisy inputs should be studied both as separate additions and as interacting partners.

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